ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 10, Issue 4, pp: (48-53), Month: October 2022 - December 2022, Available at: www.researchpublish.com

Fractional Differential Problem of Some Fractional Trigonometric Functions

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DOI: https://doi.org/10.5281/zenodo.7458073

Published Date: 19-December-2022

Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus, we study the fractional differential problem of two types of fractional trigonometric functions. Using chain rule for fractional derivatives and a new multiplication of fractional analytic functions, we can obtain the fractional derivatives of any order of these two types of fractional trigonometric functions. On the other hand, we provide some examples to illustrate our methods. In fact, our results are generalizations of those results in classical calculus.**

Keywords: **Jumarie's modified R-L fractional calculus, fractional trigonometric functions, chain rule for fractional derivatives, new multiplication, fractional analytic functions.**

I. INTRODUCTION

In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. In fact, the latest progress of fractional calculus is mainly in physics, mechanics, electrical engineering, economics, viscoelasticity, biology, control theory, and other fields [1-10]. There is no doubt that fractional calculus has become an exciting new mathematical tool to solve various problems in mathematics, science and engineering.

However, the definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus, the fractional derivatives of any order of two types of fractional trigonometric functions are obtained. A new multiplication of fractional analytic functions and chain rule for fractional derivatives play important roles in this article. Moreover, we give two examples to illustrate the application of our results. In fact, our results are natural generalizations of those results in ordinary calculus.

II. DEFINITIONS AND PROPERTIES

Firstly, the fractional calculus used in this paper and its properties are introduced below.

Definition 2.1 ([16])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{\theta_0}D_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{\theta_0}^{\theta}\frac{f(t)-f(\theta_0)}{(\theta-t)^{\alpha}}dt\ . \tag{1}$$

And the Jumarie's modified R-L α-fractional integral is defined by

$$\left({}_{\theta_0}I_{\theta}^{\alpha}\right)[f(\theta)] = \frac{1}{\Gamma(\alpha)}\int_{\theta_0}^{\theta}\frac{f(t)}{(\theta-t)^{1-\alpha}}dt\ , \tag{2}$$

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where $\Gamma(\ )$ is the gamma function. Moreover, we define $\left({}_{\theta_0}D_\theta^\alpha\right)^n[f(\theta)] = \left({}_{\theta_0}D_\theta^\alpha\right)\left({}_{\theta_0}D_\theta^\alpha\right)\cdots\left({}_{\theta_0}D_\theta^\alpha\right)[f(\theta)]$, and it is called the n-th order α-fractional derivative of $f(\theta)$, where n is any positive integer.

Proposition 2.2 ([17])**:** *If $\alpha, \beta, \theta_0, C$ are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{\theta_0}D_\theta^\alpha\right)\left[(\theta-\theta_0)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\theta-\theta_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{\theta_0}D_\theta^\alpha\right)[C] = 0. \tag{4}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** Assume that θ, θ_0, and a_k are real numbers for all k, $\theta_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha}$ on some open interval containing θ_0, then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at θ_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([19])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. If $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned} &f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(\theta-\theta_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} &f_\alpha(\theta^\alpha) \otimes g_\alpha(\theta^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5 ([19])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ be two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(\theta^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\theta-\theta_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = f_\alpha\left(g_\alpha(\theta^\alpha)\right) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(\theta^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(\theta^\alpha) = g_\alpha\left(f_\alpha(\theta^\alpha)\right) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(\theta^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.6 ([19])**:** Let $0<\alpha\leq 1$. If $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions at θ_0 satisfies

$$(f_\alpha\circ g_\alpha)(\theta^\alpha)=(g_\alpha\circ f_\alpha)(\theta^\alpha)=\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha. \tag{13}$$

Then $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([20])**:** If $0<\alpha\leq 1$, and θ is any real number. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{\theta^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes k}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(\theta^\alpha)$ is the inverse function of $E_\alpha(\theta^\alpha)$. In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k\alpha}}{\Gamma(2k\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes 2k}, \tag{15}$$

and

$$sin_\alpha(\theta^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes(2k+1)}. \tag{16}$$

Proposition 2.8 (fractional Euler's formula)**:** *Let $0<\alpha\leq 1$, θ be a real number, then*

$$E_\alpha(i\theta^\alpha)=cos_\alpha(\theta^\alpha)+isin_\alpha(\theta^\alpha). \tag{17}$$

Definition 2.9: Assume that $0<\alpha\leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(\theta^\alpha)\right)^{\otimes n}=f_\alpha(\theta^\alpha)\otimes\cdots\otimes f_\alpha(\theta^\alpha)$ is called the nth power of $f_\alpha(\theta^\alpha)$. On the other hand, if $f_\alpha(\theta^\alpha)\otimes g_\alpha(\theta^\alpha)=1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $\left(f_\alpha(\theta^\alpha)\right)^{\otimes -1}$.

Definition 2.10 ([16])**:** Let $0<\alpha\leq 1$, and p be a real number. The p-th power of the α-fractional analytic function $f_\alpha(\theta^\alpha)$ is defined by $[f_\alpha(\theta^\alpha)]^{\otimes p}=E_\alpha\left(pLn_\alpha\left(f_\alpha(\theta^\alpha)\right)\right)$.

Proposition 2.11: ([21])**:** *Let $0<\alpha\leq 1$, and θ be a real number, then*

$$[sin_\alpha(\theta^\alpha)]^{\otimes 2}+[cos_\alpha(\theta^\alpha)]^{\otimes 2}=1. \tag{18}$$

Theorem 2.12: (chain rule for fractional derivatives) ([20])**:** *If $0<\alpha\leq 1$, θ,θ_0 are real numbers, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are α-fractional analytic functions at θ_0. Then*

$$\left({}_{\theta_0}D_\theta^\alpha\right)\left[f_\alpha\left(g_\alpha(\theta^\alpha)\right)\right]=\left({}_{\theta_0}D_\theta^\alpha\right)[f_\alpha(\theta^\alpha)]\left(g_\alpha(\theta^\alpha)\right)\otimes\left({}_{\theta_0}D_\theta^\alpha\right)[g_\alpha(\theta^\alpha)]. \tag{19}$$

Definition 2.13: Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, a,b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z is denoted by $\mathrm{Im}(z)$.

Definition 2.14: If $0<\alpha\leq 1$, and $x_\alpha(\theta^\alpha)$, $y_\alpha(\theta^\alpha)$ are real α-fractional analytic functions, then $z_\alpha(\theta^\alpha)=x_\alpha(\theta^\alpha)+iy_\alpha(\theta^\alpha)$ is a complex α-fractional analytic function. We define

$$|z_\alpha(\theta^\alpha)|_\otimes=\left[[x_\alpha(\theta^\alpha)]^{\otimes 2}+[y_\alpha(\theta^\alpha)]^{\otimes 2}\right]^{\otimes\frac{1}{2}}. \tag{20}$$

Definition 2.15: The smallest positive real number T_α such that $E_\alpha(iT_\alpha)=1$, is called the period of $E_\alpha(i\theta^\alpha)$.

III. RESULTS AND EXAMPLES

In this section, we introduce some results including the fractional derivatives of any order of two types of fractional trigonometric functions. In addition, some examples are provided to illustrate the application of our results.

Proposition 3.1: *Suppose that $0<\alpha\leq 1$, $z_\alpha(\theta^\alpha)=x_\alpha(\theta^\alpha)+iy_\alpha(\theta^\alpha)$ is a complex α-fractional analytic function and $|z_\alpha(\theta^\alpha)|_\otimes<1$. Then*

$$Ln_\alpha\left(1+z_\alpha(\theta^\alpha)\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}[z_\alpha(\theta^\alpha)]^{\otimes k}. \tag{21}$$

Proof Since $|z_\alpha(\theta^\alpha)|_\otimes < 1$, by chain rule for fractional derivatives,

$$\left({}_0D_\theta^\alpha\right)\left[Ln_\alpha\left(1+z_\alpha(\theta^\alpha)\right)\right]$$
$$=\left(1+z_\alpha(\theta^\alpha)\right)^{\otimes -1}\otimes\left({}_0D_\theta^\alpha\right)[z_\alpha(\theta^\alpha)]$$
$$=\sum_{k=1}^{\infty}(-1)^{k-1}\,[z_\alpha(\theta^\alpha)]^{\otimes k}\otimes\left({}_0D_\theta^\alpha\right)[z_\alpha(\theta^\alpha)]. \tag{22}$$

It follows that

$$Ln_\alpha\left(1+z_\alpha(\theta^\alpha)\right)$$
$$=\left({}_0I_\theta^\alpha\right)\left[\sum_{k=1}^{\infty}(-1)^{k-1}\,[z_\alpha(\theta^\alpha)]^{\otimes k}\otimes\left({}_0D_\theta^\alpha\right)[z_\alpha(\theta^\alpha)]\right]$$
$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}[z_\alpha(\theta^\alpha)]^{\otimes k}\,.$$

Q.e.d.

Proposition 3.2: *Let* $0<\alpha\le 1$ *and* r *be a real number,* $|r|<1$. *Then*

$$Ln_\alpha\left(1+rE_\alpha(i\theta^\alpha)\right)=\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(\theta^\alpha)+r^2)+iarctan\left(rsin_\alpha(\theta^\alpha)\otimes\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right). \tag{23}$$

Proof $Ln_\alpha\left(1+rE_\alpha(i\theta^\alpha)\right)$

$$=Ln_\alpha\left(1+rcos_\alpha(\theta^\alpha)+irsin_\alpha(\theta^\alpha)\right)\quad\text{(by fractional Euler's formula)}$$

$$=Ln_\alpha\left[\begin{array}{c}\left[\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes 2}+\left(rsin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes\frac{1}{2}}\\ \otimes\left[\left[\left(1+rcos_\alpha(\theta^\alpha)\right)+i\left(rsin_\alpha(\theta^\alpha)\right)\right]\otimes\left[\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes 2}+\left(rsin_\alpha(\theta^\alpha)\right)^{\otimes 2}\right]^{\otimes -\frac{1}{2}}\right]\end{array}\right]$$

$$=Ln_\alpha\left[\begin{array}{c}[(1+2rcos_\alpha(\theta^\alpha)+r^2)]^{\otimes\frac{1}{2}}\\ \otimes\left[\left[\left(1+rcos_\alpha(\theta^\alpha)\right)+i\left(rsin_\alpha(\theta^\alpha)\right)\right]\otimes[(1+2rcos_\alpha(\theta^\alpha)+r^2)]^{\otimes -\frac{1}{2}}\right]\end{array}\right]$$

$$=Ln_\alpha\left[[(1+2rcos_\alpha(\theta^\alpha)+r^2)]^{\otimes\frac{1}{2}}\right]$$
$$+Ln_\alpha\left[\left[\left[\left(1+rcos_\alpha(\theta^\alpha)\right)+i\left(rsin_\alpha(\theta^\alpha)\right)\right]\otimes[(1+2rcos_\alpha(\theta^\alpha)+r^2)]^{\otimes -\frac{1}{2}}\right]\right]$$

$$=\frac{1}{2}Ln_\alpha[(1+2rcos_\alpha(\theta^\alpha)+r^2)]+Ln_\alpha\left[E_\alpha\left[iarctan\left(rsin_\alpha(\theta^\alpha)\otimes\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)\right]\right]$$

$$=\frac{1}{2}Ln_\alpha(1+2rcos_\alpha(\theta^\alpha)+r^2)+iarctan\left(rsin_\alpha(\theta^\alpha)\otimes\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right).$$

Q.e.d.

Proposition 3.3: *If* $0<\alpha\le 1$ *and* $|r|<1$. *Then*

$$Ln_\alpha(1+2rcos_\alpha(\theta^\alpha)+r^2)=2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}r^k cos_\alpha(k\theta^\alpha), \tag{24}$$

and

$$arctan\left(rsin_\alpha(\theta^\alpha)\otimes\left(1+rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}r^k sin_\alpha(k\theta^\alpha). \tag{25}$$

Proof $Ln_\alpha(1+2rcos_\alpha(\theta^\alpha)+r^2)$

$$=2\cdot\mathrm{Re}\left[Ln_\alpha\left(1+rE_\alpha(i\theta^\alpha)\right)\right]$$
$$=2\cdot\mathrm{Re}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(rE_\alpha(i\theta^\alpha)\right)^{\otimes k}\right]$$

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
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$$= 2 \cdot \mathrm{Re}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k E_\alpha(ik\theta^\alpha)\right]$$

$$= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k cos_\alpha(k\theta^\alpha) .$$

On the other hand,

$$arctan\left(rsin_\alpha(\theta^\alpha) \otimes \left(1 + rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)$$

$$= \mathrm{Im}\left[Ln_\alpha\left(1 + rE_\alpha(i\theta^\alpha)\right)\right]$$

$$= \mathrm{Im}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k E_\alpha(ik\theta^\alpha)\right]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k sin_\alpha(k\theta^\alpha) .$$

Q.e.d.

Theorem 3.4: *Suppose that* $0 < \alpha \leq 1$, $|r| < 1$, *and* n *is any positive integer. Then*

$$\left({}_0D_\theta^\alpha\right)^n [Ln_\alpha(1 + 2rcos_\alpha(\theta^\alpha) + r^2)] = 2 \sum_{k=1}^{\infty} k^{n-1}(-1)^{k-1} r^k cos_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right), \tag{26}$$

and

$$\left({}_0D_\theta^\alpha\right)^n \left[arctan\left(rsin_\alpha(\theta^\alpha) \otimes \left(1 + rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)\right] = \sum_{k=1}^{\infty} k^{n-1}(-1)^{k-1} r^k sin_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right). \tag{27}$$

Proof

$$\left({}_0D_\theta^\alpha\right)^n [Ln_\alpha(1 + 2rcos_\alpha(\theta^\alpha) + r^2)]$$

$$= \left({}_0D_\theta^\alpha\right)^n \left[2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k cos_\alpha(k\theta^\alpha)\right]$$

$$= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k \left({}_0D_\theta^\alpha\right)^n [cos_\alpha(k\theta^\alpha)]$$

$$= 2 \sum_{k=1}^{\infty} k^{n-1}(-1)^{k-1} r^k cos_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right).$$

And

$$\left({}_0D_\theta^\alpha\right)^n \left[arctan\left(rsin_\alpha(\theta^\alpha) \otimes \left(1 + rcos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)\right]$$

$$= \left({}_0D_\theta^\alpha\right)^n \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k sin_\alpha(k\theta^\alpha)\right]$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} r^k \left({}_0D_\theta^\alpha\right)^n [sin_\alpha(k\theta^\alpha)]$$

$$= \sum_{k=1}^{\infty} k^{n-1}(-1)^{k-1} r^k sin_\alpha\left(k\theta^\alpha + n \cdot \frac{T_\alpha}{4}\right).$$

Q.e.d.

Example 3.5: Let $0 < \alpha \leq 1$.Then

$$\left({}_0D_\theta^\alpha\right)^{23} \left[Ln_\alpha\left(\frac{10}{9} + \frac{2}{3} cos_\alpha(\theta^\alpha)\right)\right]$$

$$= 2 \sum_{k=1}^{\infty} k^{22}(-1)^{k-1} \left(\frac{1}{3}\right)^k cos_\alpha\left(k\theta^\alpha + 23 \cdot \frac{T_\alpha}{4}\right)$$

$$= 2 \sum_{k=1}^{\infty} k^{22}(-1)^{k-1} \left(\frac{1}{3}\right)^k sin_\alpha(k\theta^\alpha) . \tag{28}$$

Example 3.6: Assume that $0 < \alpha \leq 1$. Then

$$\left({}_0D_\theta^\alpha\right)^{17} \left[arctan\left(\frac{1}{4} sin_\alpha(\theta^\alpha) \otimes \left(1 + \frac{1}{4} cos_\alpha(\theta^\alpha)\right)^{\otimes -1}\right)\right]$$

$$= \sum_{k=1}^{\infty} k^{16}(-1)^{k-1} \left(\frac{1}{4}\right)^k sin_\alpha\left(k\theta^\alpha + 17 \cdot \frac{T_\alpha}{4}\right)$$

$$= \sum_{k=1}^{\infty} k^{16}(-1)^{k-1} \left(\frac{1}{4}\right)^k cos_\alpha(k\theta^\alpha) . \tag{29}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, we find the fractional derivatives of any order of two types of fractional trigonometric functions. Chain rule for fractional derivatives and a new multiplication of fractional analytic functions play important roles in this paper. Furthermore, we provide some examples to illustrate how to use our results to obtain fractional derivatives of any order of these two types of fractional trigonometric functions. In fact, the results we obtained are generalizations of those results in traditional calculus. In the future, we will continue to use our methods to study the problems in applied mathematics and fractional differential equations.

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